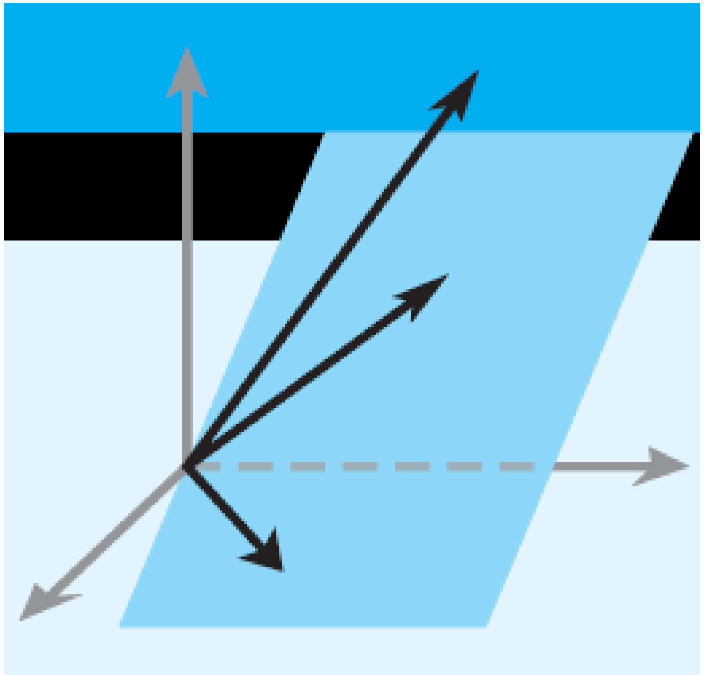
Elementary Linear Algebra



Chapter 3

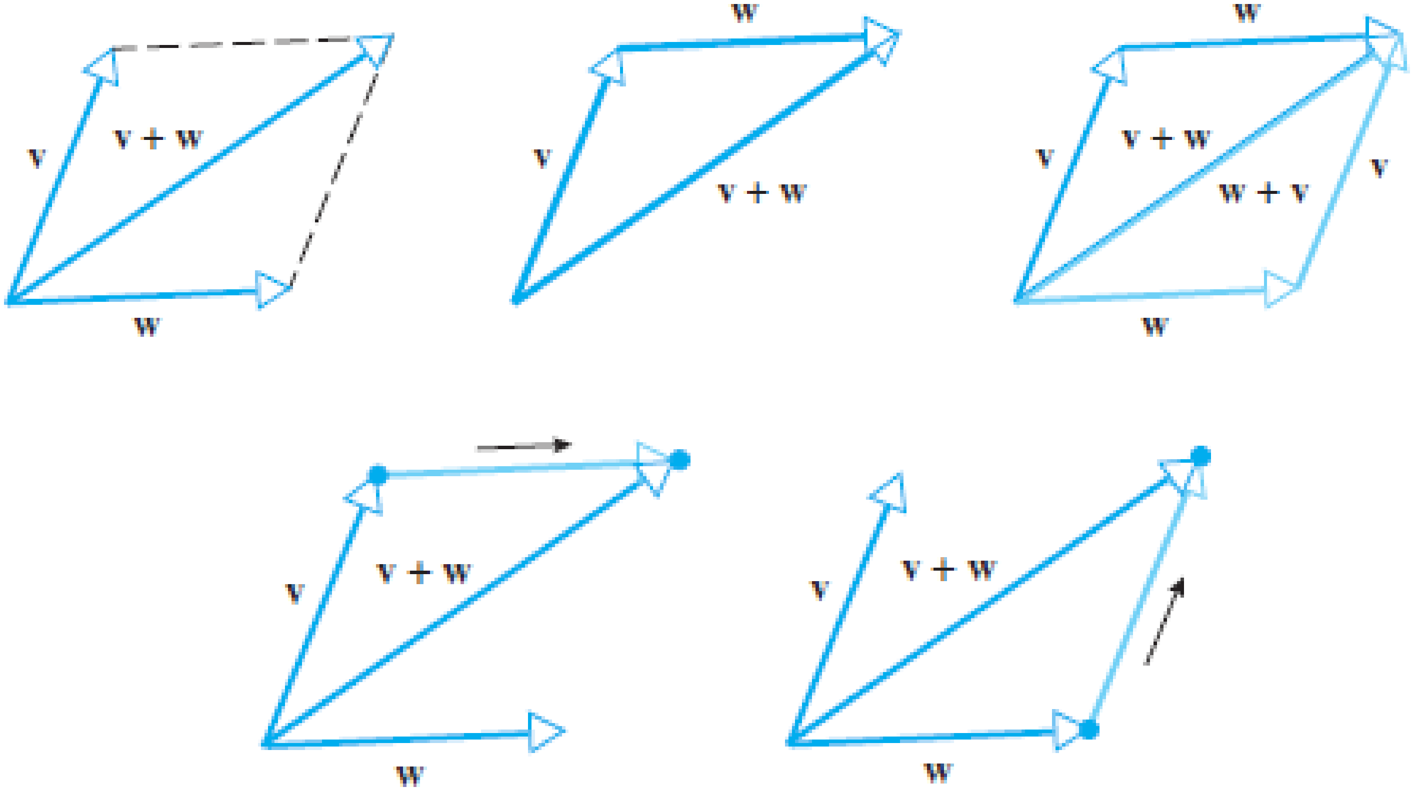
Howard Anton

Chapter 3
Euclidean Vector Spaces

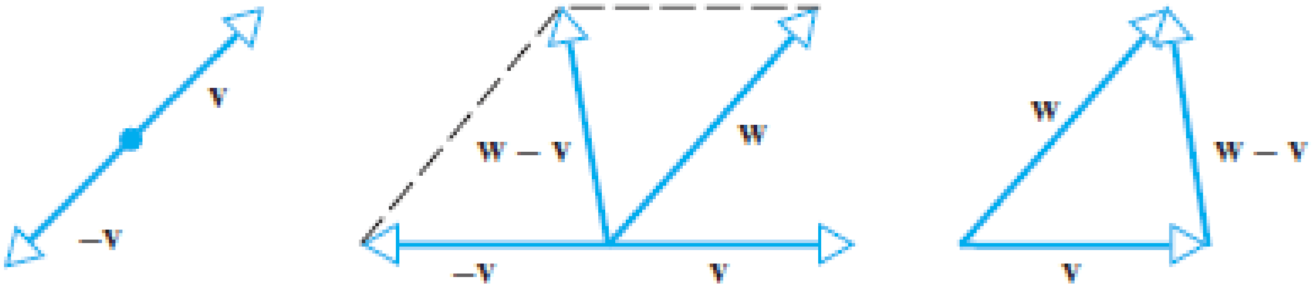
- 3.1 Vectors in 2-Space, 3-Space, and n-Space
- 3.2 Norm, Dot Product, and Distance in R^n
- 3.3 Orthogonality
- 3.4 The Geometry of Linear Systems
- 3.5 Cross Product

Section 3.1 Vectors

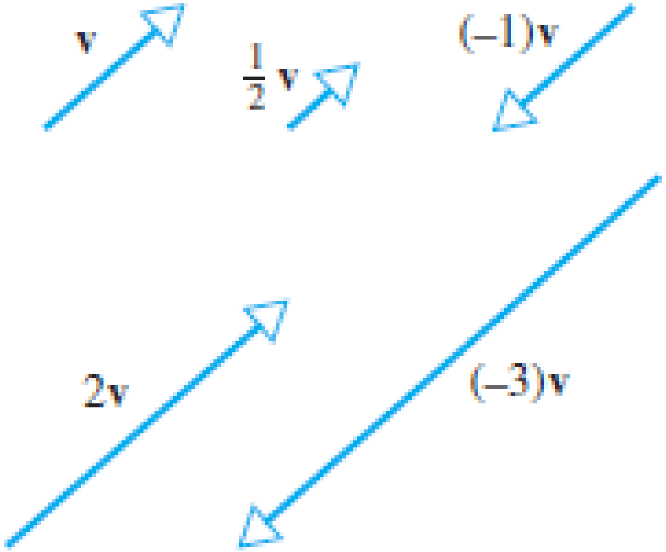
Addition of vectors by the parallelogram or triangle rules



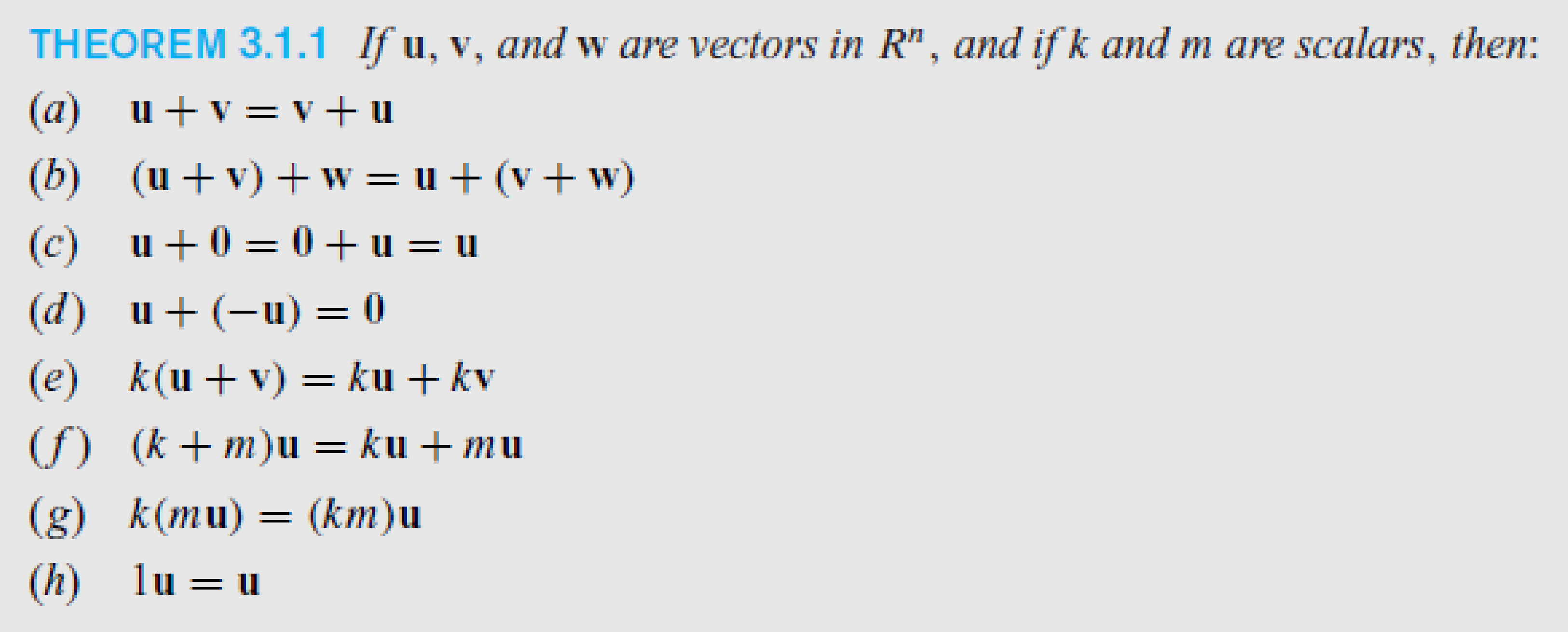
Subtraction:



Scalar Multiplication:



Properties of Vectors



THEOREM 3.1.1 *If* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are vectors in* R^n, *and if* k *and* m *are scalars, then:*

(*a*) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$

(*b*) $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$

(*c*) $\mathbf{u}+\mathbf{0}=\mathbf{0}+\mathbf{u}=\mathbf{u}$

(*d*) $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$

(*e*) $k(\mathbf{u}+\mathbf{v})=k\mathbf{u}+k\mathbf{v}$

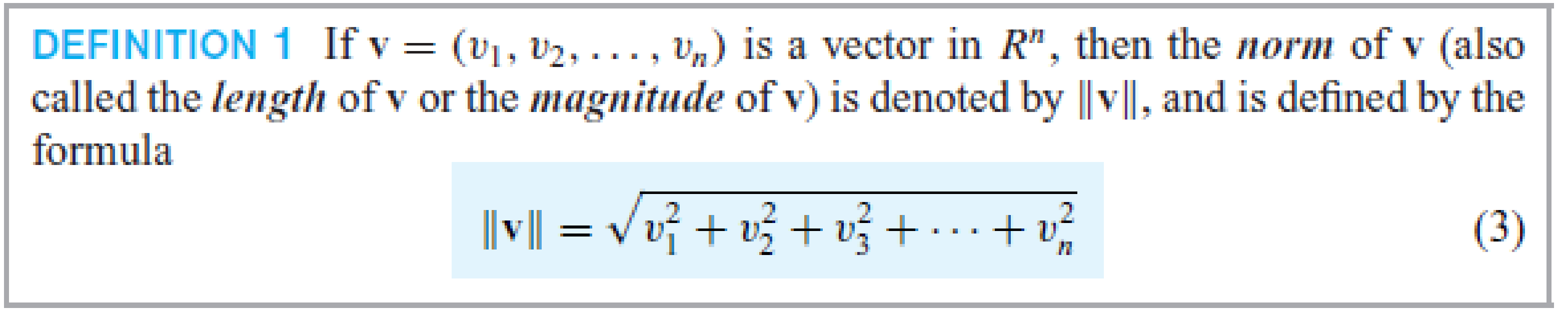
(*f*) $(k+m)\mathbf{u}=k\mathbf{u}+m\mathbf{u}$

(*g*) $k(m\mathbf{u})=(km)\mathbf{u}$

(*h*) $1\mathbf{u}=\mathbf{u}$

Section 3.2 Norm, Dot Product, and Distance in R^n

Norm:



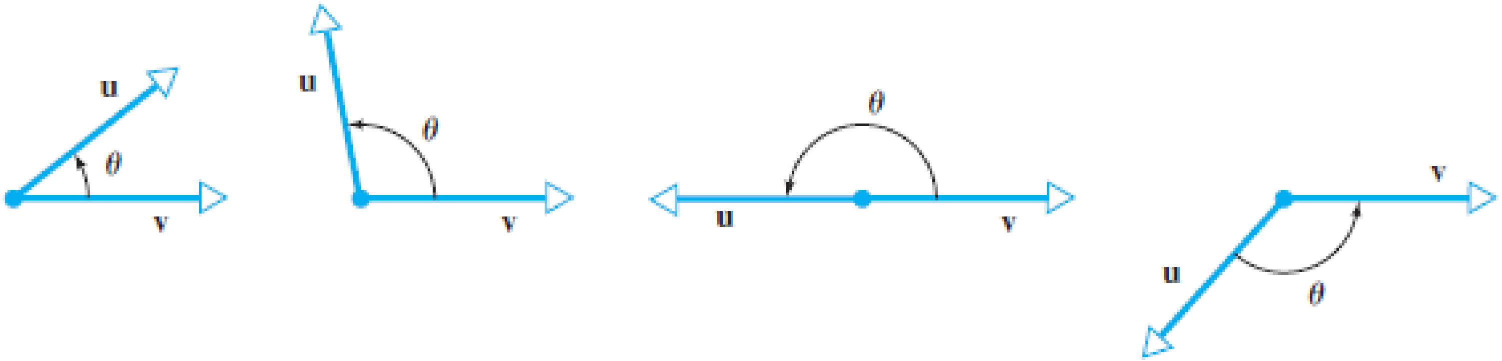
DEFINITION 1 If $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ is a vector in R^n, then the *norm* of $\mathbf{v}$ (also called the *length* of $\mathbf{v}$ or the *magnitude* of $\mathbf{v}$) is denoted by $\|\mathbf{v}\|$, and is defined by the formula

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2 + \cdots + v_n^2} \tag{3}$$

Unit Vectors:

$$\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v}$$

The Dot Product



The angle θ between $\mathbf{u}$ and $\mathbf{v}$ satisfies $0 \leq \theta \leq \pi$.

DEFINITION 3 If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors in R^2 or R^3, and if θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then the *dot product* (also called the *Euclidean inner product*) of $\mathbf{u}$ and $\mathbf{v}$ is denoted by $\mathbf{u} \cdot \mathbf{v}$ and is defined as

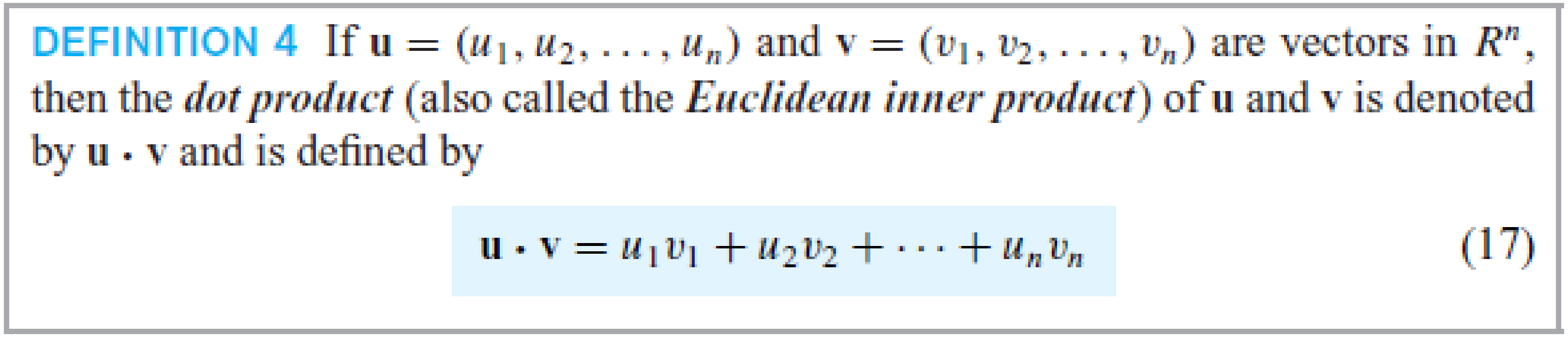
$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos\theta \tag{12}$$

If $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$, then we define $\mathbf{u} \cdot \mathbf{v}$ to be 0.

The sign of the dot product reveals information about the angle θ that we can obtain by rewriting Formula (12) as

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} \tag{13}$$

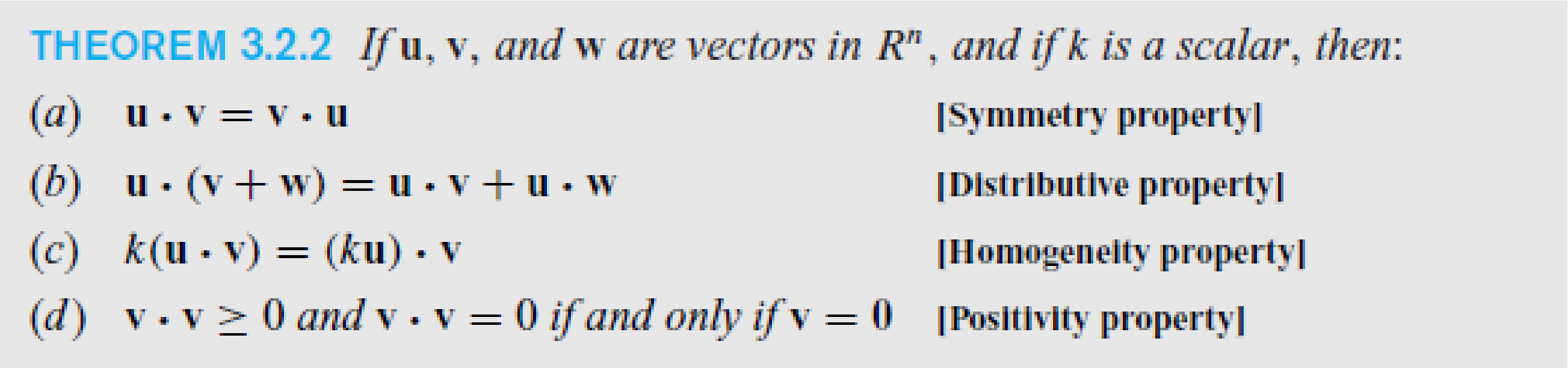
The Dot Product



DEFINITION 4 If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are vectors in R^n, then the *dot product* (also called the *Euclidean inner product*) of **u** and **v** is denoted by $\mathbf{u} \cdot \mathbf{v}$ and is defined by

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n \tag{17}$$

Properties of the Dot Product



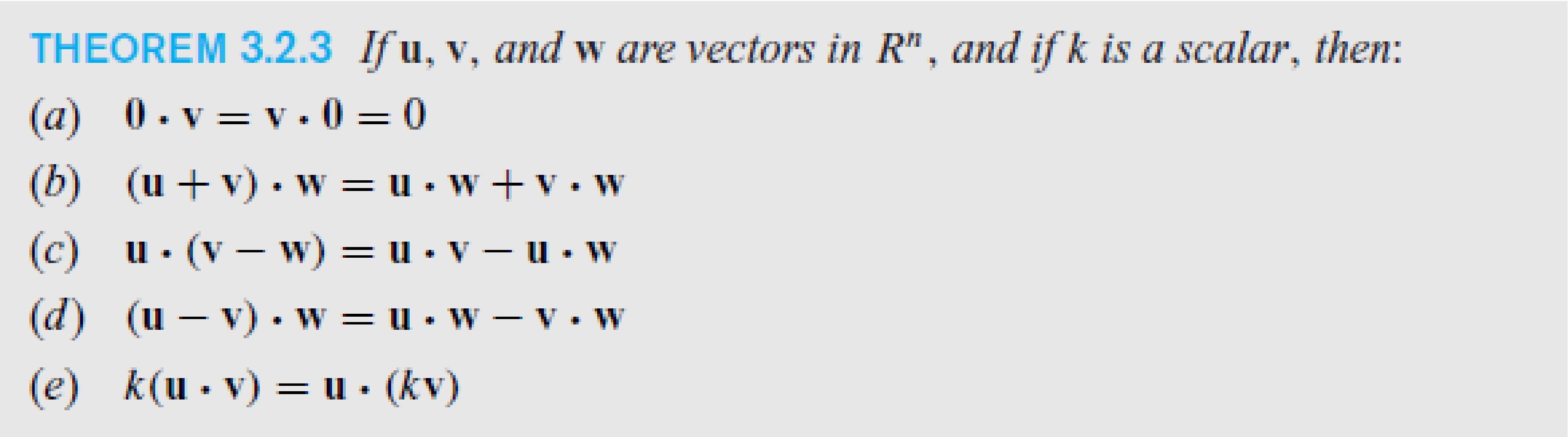
THEOREM 3.2.2 *If* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are vectors in* R^n, *and if* k *is a scalar, then:*

(*a*) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ **[Symmetry property]**

(*b*) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ **[Distributive property]**

(*c*) $k(\mathbf{u} \cdot \mathbf{v}) = (k\mathbf{u}) \cdot \mathbf{v}$ **[Homogeneity property]**

(*d*) $\mathbf{v} \cdot \mathbf{v} \geq 0$ *and* $\mathbf{v} \cdot \mathbf{v} = 0$ *if and only if* $\mathbf{v} = \mathbf{0}$ **[Positivity property]**



THEOREM 3.2.3 *If* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are vectors in* R^n, *and if* k *is a scalar, then:*

(*a*) $\mathbf{0} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{0} = 0$

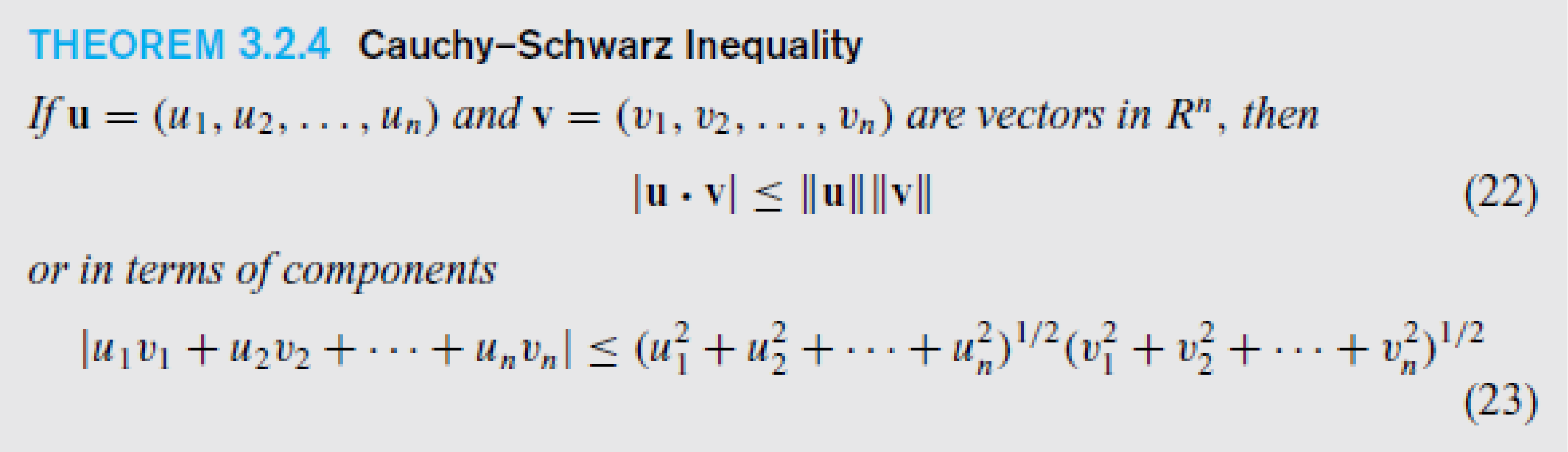
(*b*) $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$

(*c*) $\mathbf{u} \cdot (\mathbf{v} - \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{w}$

(*d*) $(\mathbf{u} - \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} - \mathbf{v} \cdot \mathbf{w}$

(*e*) $k(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (k\mathbf{v})$

Cauchy-Schwarz Inequality



THEOREM 3.2.4 Cauchy–Schwarz Inequality

If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ *and* $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ *are vectors in* R^n, *then*

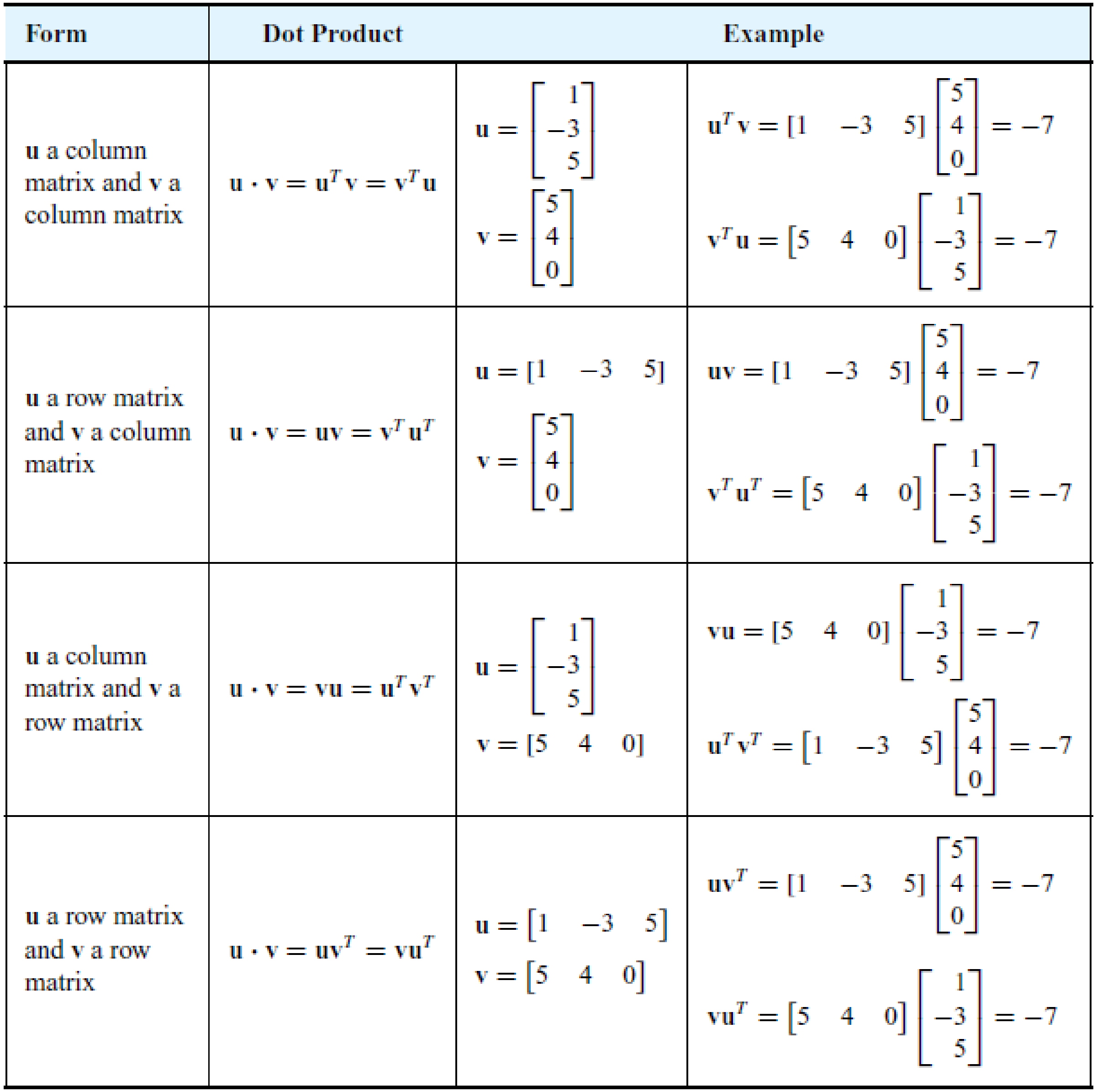
$$|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\| \, \|\mathbf{v}\| \tag{22}$$

or in terms of components

$$|u_1 v_1 + u_2 v_2 + \cdots + u_n v_n| \leq (u_1^2 + u_2^2 + \cdots + u_n^2)^{1/2} (v_1^2 + v_2^2 + \cdots + v_n^2)^{1/2} \tag{23}$$

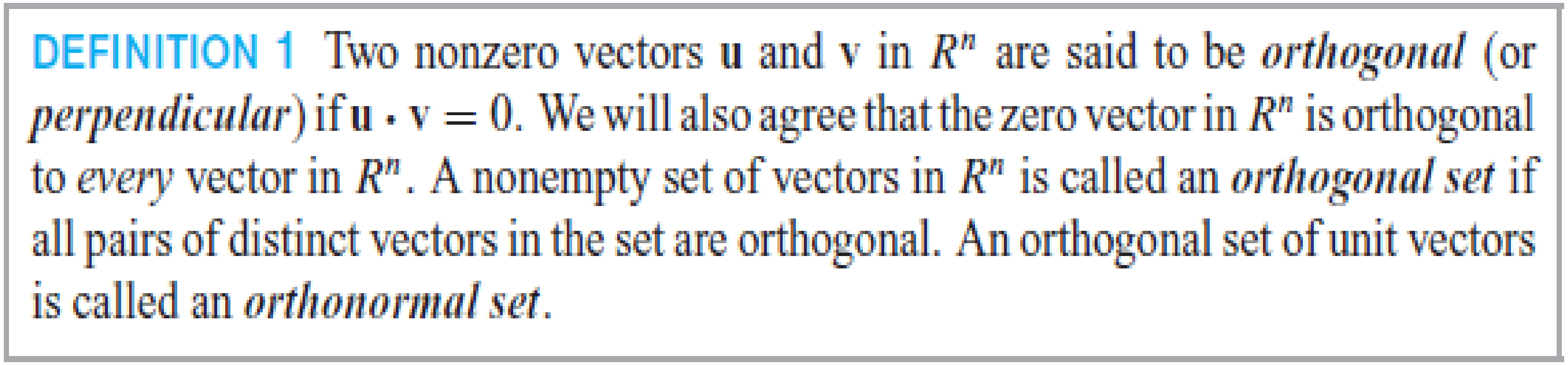
Dot Products and Matrices

Table 1



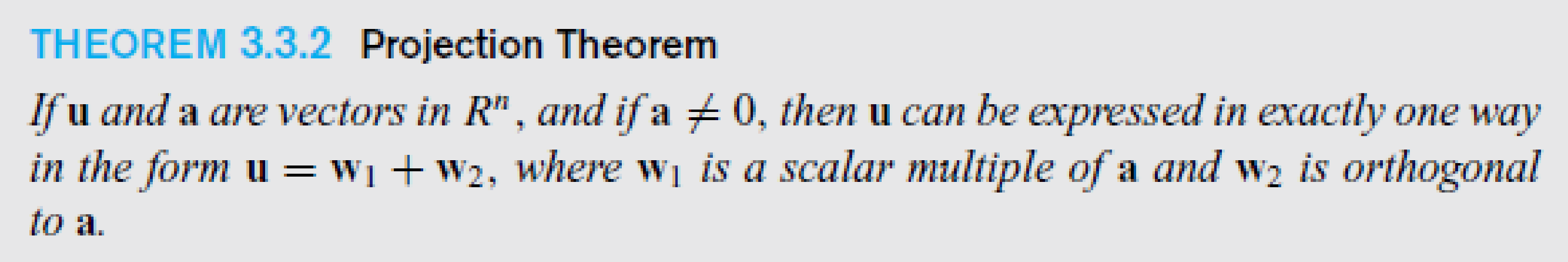
Form	Dot Product	Example	
u a column matrix and **v** a column matrix	$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T\mathbf{v} = \mathbf{v}^T\mathbf{u}$	$\mathbf{u} = \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix}$ $\mathbf{v} = \begin{bmatrix} 5 \\ 4 \\ 0 \end{bmatrix}$	$\mathbf{u}^T\mathbf{v} = [1 \quad -3 \quad 5] \begin{bmatrix} 5 \\ 4 \\ 0 \end{bmatrix} = -7$ $\mathbf{v}^T\mathbf{u} = [5 \quad 4 \quad 0] \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix} = -7$
u a row matrix and **v** a column matrix	$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}\mathbf{v} = \mathbf{v}^T\mathbf{u}^T$	$\mathbf{u} = [1 \quad -3 \quad 5]$ $\mathbf{v} = \begin{bmatrix} 5 \\ 4 \\ 0 \end{bmatrix}$	$\mathbf{u}\mathbf{v} = [1 \quad -3 \quad 5] \begin{bmatrix} 5 \\ 4 \\ 0 \end{bmatrix} = -7$ $\mathbf{v}^T\mathbf{u}^T = [5 \quad 4 \quad 0] \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix} = -7$
u a column matrix and **v** a row matrix	$\mathbf{u} \cdot \mathbf{v} = \mathbf{v}\mathbf{u} = \mathbf{u}^T\mathbf{v}^T$	$\mathbf{u} = \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix}$ $\mathbf{v} = [5 \quad 4 \quad 0]$	$\mathbf{v}\mathbf{u} = [5 \quad 4 \quad 0] \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix} = -7$ $\mathbf{u}^T\mathbf{v}^T = [1 \quad -3 \quad 5] \begin{bmatrix} 5 \\ 4 \\ 0 \end{bmatrix} = -7$
u a row matrix and **v** a row matrix	$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}\mathbf{v}^T = \mathbf{v}\mathbf{u}^T$	$\mathbf{u} = [1 \quad -3 \quad 5]$ $\mathbf{v} = [5 \quad 4 \quad 0]$	$\mathbf{u}\mathbf{v}^T = [1 \quad -3 \quad 5] \begin{bmatrix} 5 \\ 4 \\ 0 \end{bmatrix} = -7$ $\mathbf{v}\mathbf{u}^T = [5 \quad 4 \quad 0] \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix} = -7$

Section 3.3 Orthogonality



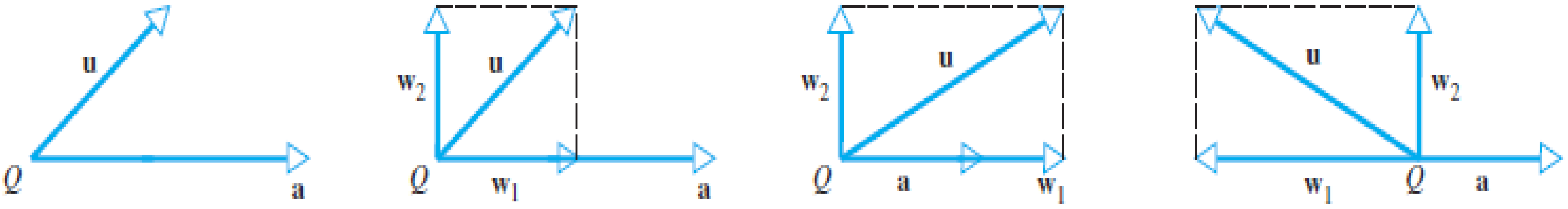
DEFINITION 1 Two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in R^n are said to be *orthogonal* (or *perpendicular*) if $\mathbf{u} \cdot \mathbf{v} = 0$. We will also agree that the zero vector in R^n is orthogonal to *every* vector in R^n. A nonempty set of vectors in R^n is called an *orthogonal set* if all pairs of distinct vectors in the set are orthogonal. An orthogonal set of unit vectors is called an *orthonormal set*.

Orthogonal Projections

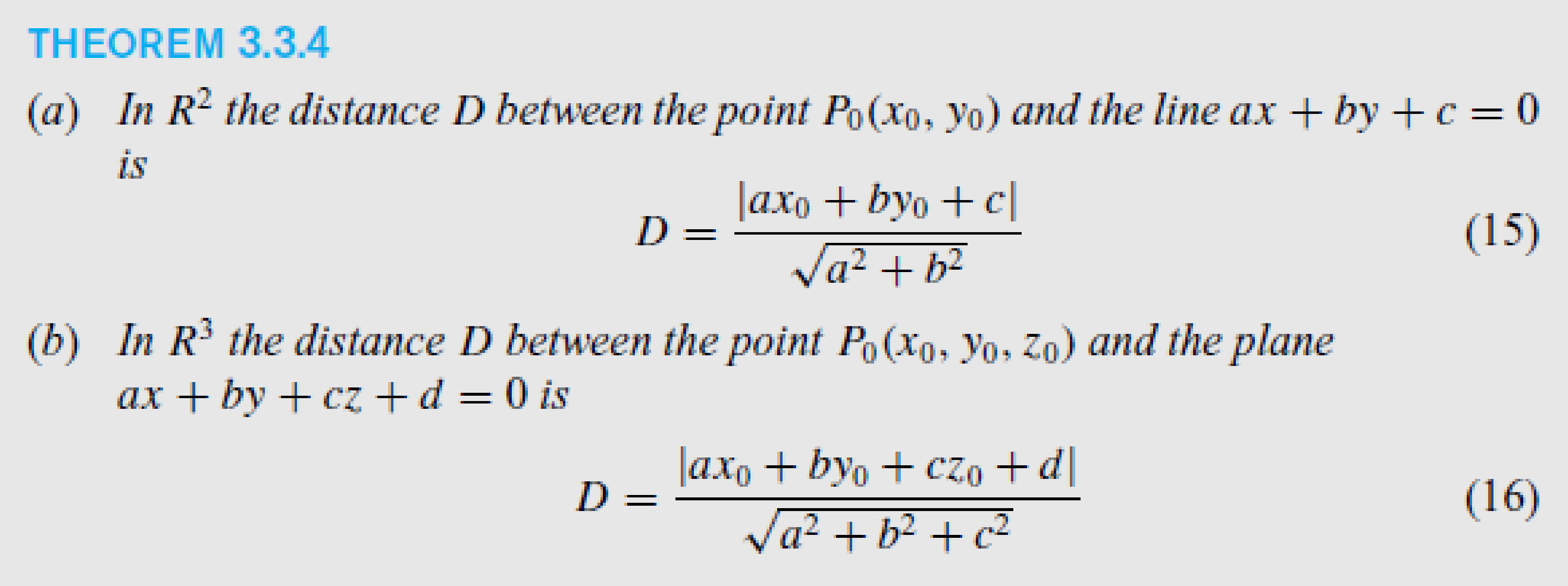


THEOREM 3.3.2 Projection Theorem

If **u** *and* **a** *are vectors in* R^n, *and if* $\mathbf{a} \neq 0$, *then* **u** *can be expressed in exactly one way in the form* $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$, *where* $\mathbf{w}_1$ *is a scalar multiple of* **a** *and* $\mathbf{w}_2$ *is orthogonal to* **a**.



Point-line and point-plane Distance formulas



THEOREM 3.3.4

(*a*) *In* R^2 *the distance* D *between the point* $P_0(x_0, y_0)$ *and the line* $ax + by + c = 0$ *is*

$$D = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}} \tag{15}$$

(*b*) *In* R^3 *the distance* D *between the point* $P_0(x_0, y_0, z_0)$ *and the plane* $ax + by + cz + d = 0$ *is*

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}} \tag{16}$$

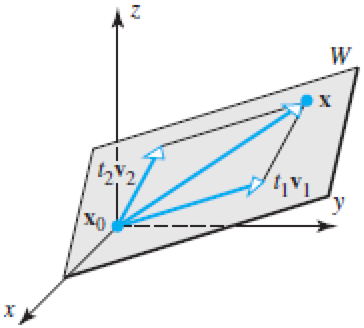
Section 3.4
The Geometry of Linear Systems

THEOREM 3.4.1 *Let L be the line in R^2 or R^3 that contains the point $\mathbf{x}_0$ and is parallel to the nonzero vector $\mathbf{v}$. Then the equation of the line through $\mathbf{x}_0$ that is parallel to $\mathbf{v}$ is*

$$\mathbf{x} = \mathbf{x}_0 + t\mathbf{v} \tag{1}$$

If $\mathbf{x}_0 = \mathbf{0}$, then the line passes through the origin and the equation has the form

$$\mathbf{x} = t\mathbf{v} \tag{2}$$

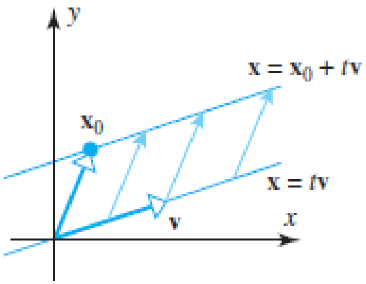


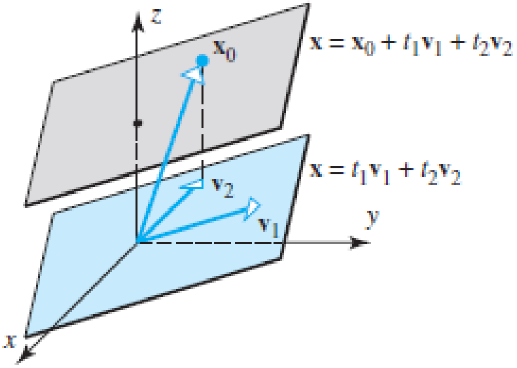
THEOREM 3.4.2 *Let W be the plane in R^3 that contains the point $\mathbf{x}_0$ and is parallel to the noncollinear vectors $\mathbf{v}_1$ and $\mathbf{v}_2$. Then an equation of the plane through $\mathbf{x}_0$ that is parallel to $\mathbf{v}_1$ and $\mathbf{v}_2$ is given by*

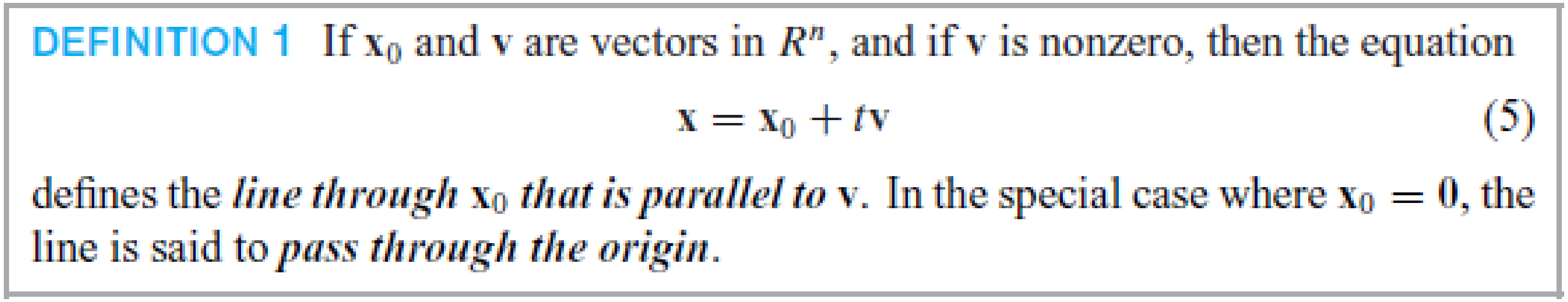
$$\mathbf{x} = \mathbf{x}_0 + t_1\mathbf{v}_1 + t_2\mathbf{v}_2 \tag{3}$$

If $\mathbf{x}_0 = \mathbf{0}$, then the plane passes through the origin and the equation has the form

$$\mathbf{x} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2 \tag{4}$$



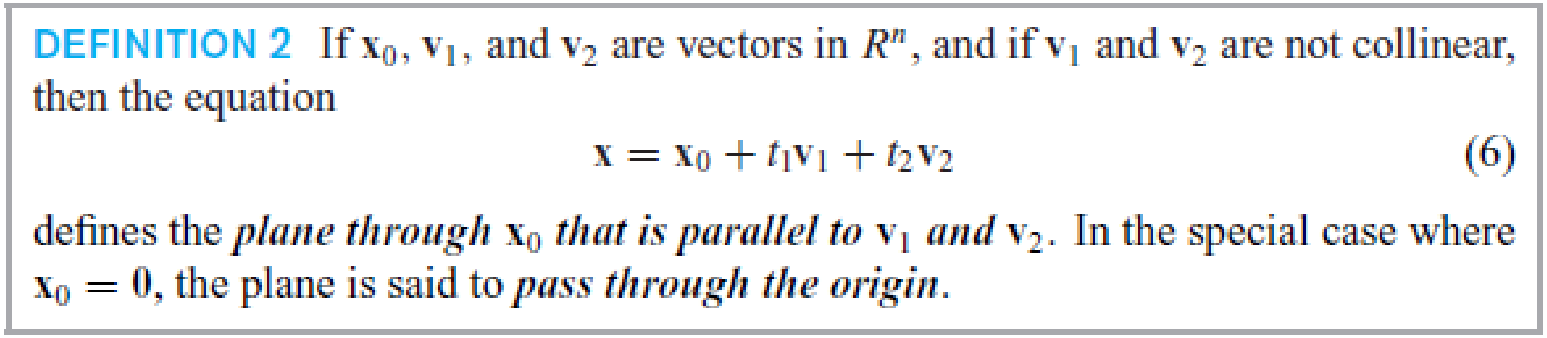




DEFINITION 1 If $\mathbf{x}_0$ and $\mathbf{v}$ are vectors in R^n, and if $\mathbf{v}$ is nonzero, then the equation

$$\mathbf{x} = \mathbf{x}_0 + t\mathbf{v} \tag{5}$$

defines the ***line through*** $\mathbf{x}_0$ ***that is parallel to*** $\mathbf{v}$. In the special case where $\mathbf{x}_0 = \mathbf{0}$, the line is said to ***pass through the origin***.

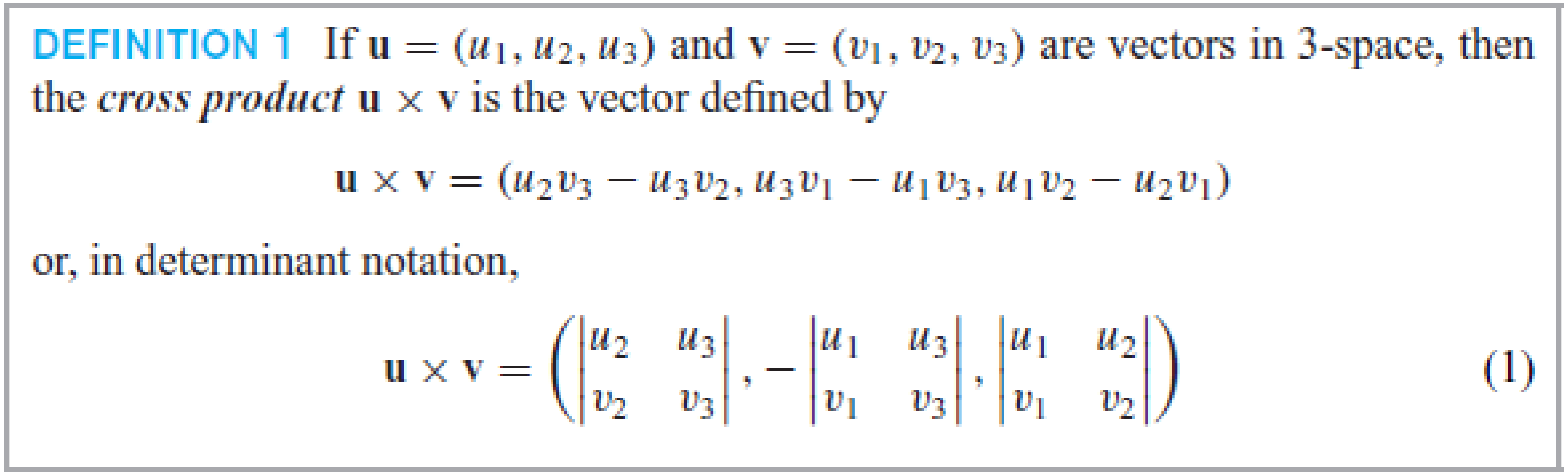


DEFINITION 2 If $\mathbf{x}_0$, $\mathbf{v}_1$, and $\mathbf{v}_2$ are vectors in R^n, and if $\mathbf{v}_1$ and $\mathbf{v}_2$ are not collinear, then the equation

$$\mathbf{x} = \mathbf{x}_0 + t_1\mathbf{v}_1 + t_2\mathbf{v}_2 \tag{6}$$

defines the ***plane through*** $\mathbf{x}_0$ ***that is parallel to*** $\mathbf{v}_1$ ***and*** $\mathbf{v}_2$. In the special case where $\mathbf{x}_0 = \mathbf{0}$, the plane is said to ***pass through the origin***.

Section 3.5 Cross Product



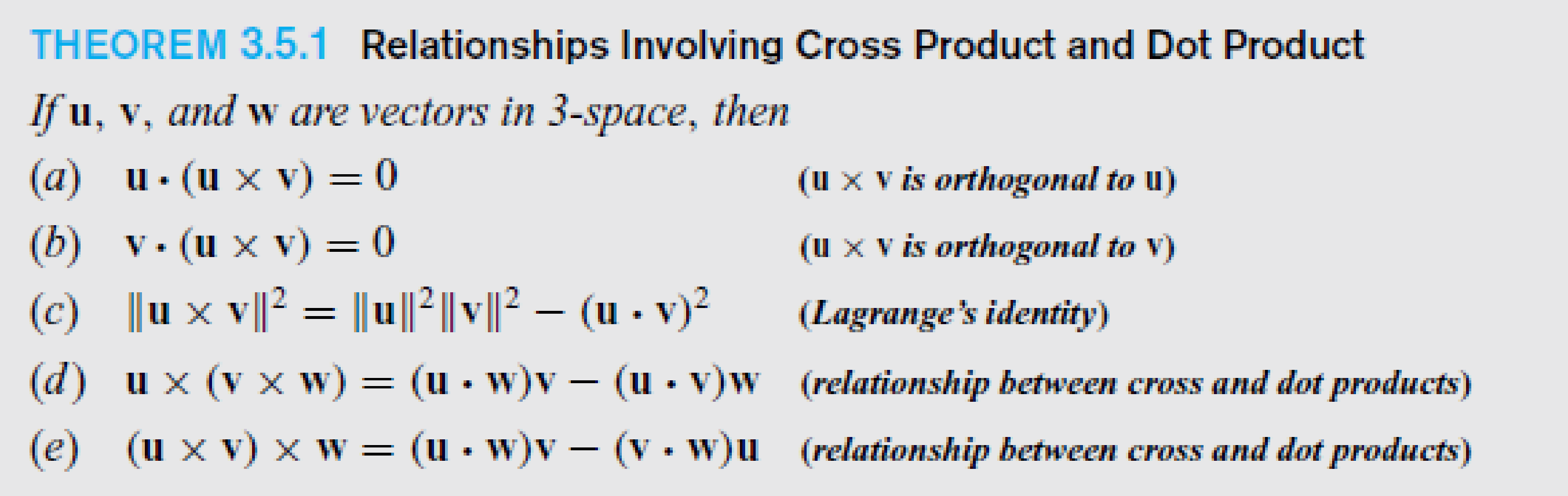
DEFINITION 1 If $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ are vectors in 3-space, then the *cross product* $\mathbf{u} \times \mathbf{v}$ is the vector defined by

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1)$$

or, in determinant notation,

$$\mathbf{u} \times \mathbf{v} = \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right) \tag{1}$$

Cross Products and Dot Products



THEOREM 3.5.1 Relationships Involving Cross Product and Dot Product

If $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are vectors in 3-space, then*

(*a*) $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$ ($\mathbf{u} \times \mathbf{v}$ *is orthogonal to* $\mathbf{u}$)

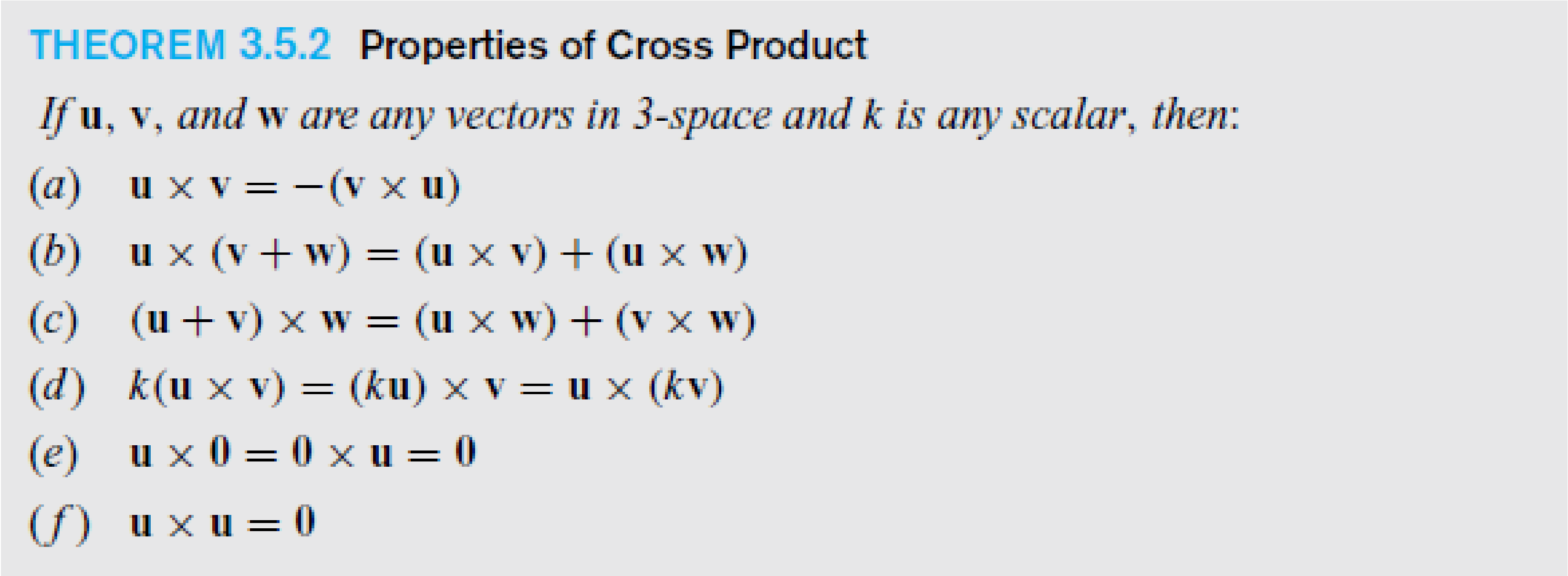
(*b*) $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$ ($\mathbf{u} \times \mathbf{v}$ *is orthogonal to* $\mathbf{v}$)

(*c*) $\|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2$ (*Lagrange's identity*)

(*d*) $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$ (*relationship between cross and dot products*)

(*e*) $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u}$ (*relationship between cross and dot products*)

Properties of Cross Product



THEOREM 3.5.2 Properties of Cross Product

If $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are any vectors in 3-space and* k *is any scalar, then:*

(*a*) $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$

(*b*) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$

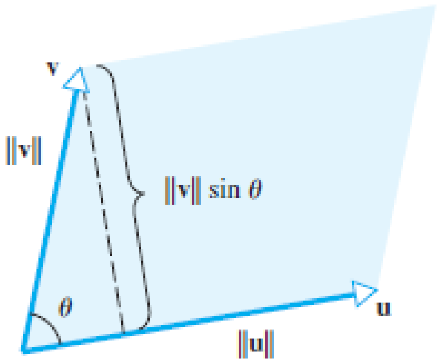
(*c*) $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w})$

(*d*) $k(\mathbf{u} \times \mathbf{v}) = (k\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (k\mathbf{v})$

(*e*) $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$

(*f*) $\mathbf{u} \times \mathbf{u} = \mathbf{0}$

Geometry of the Cross Product

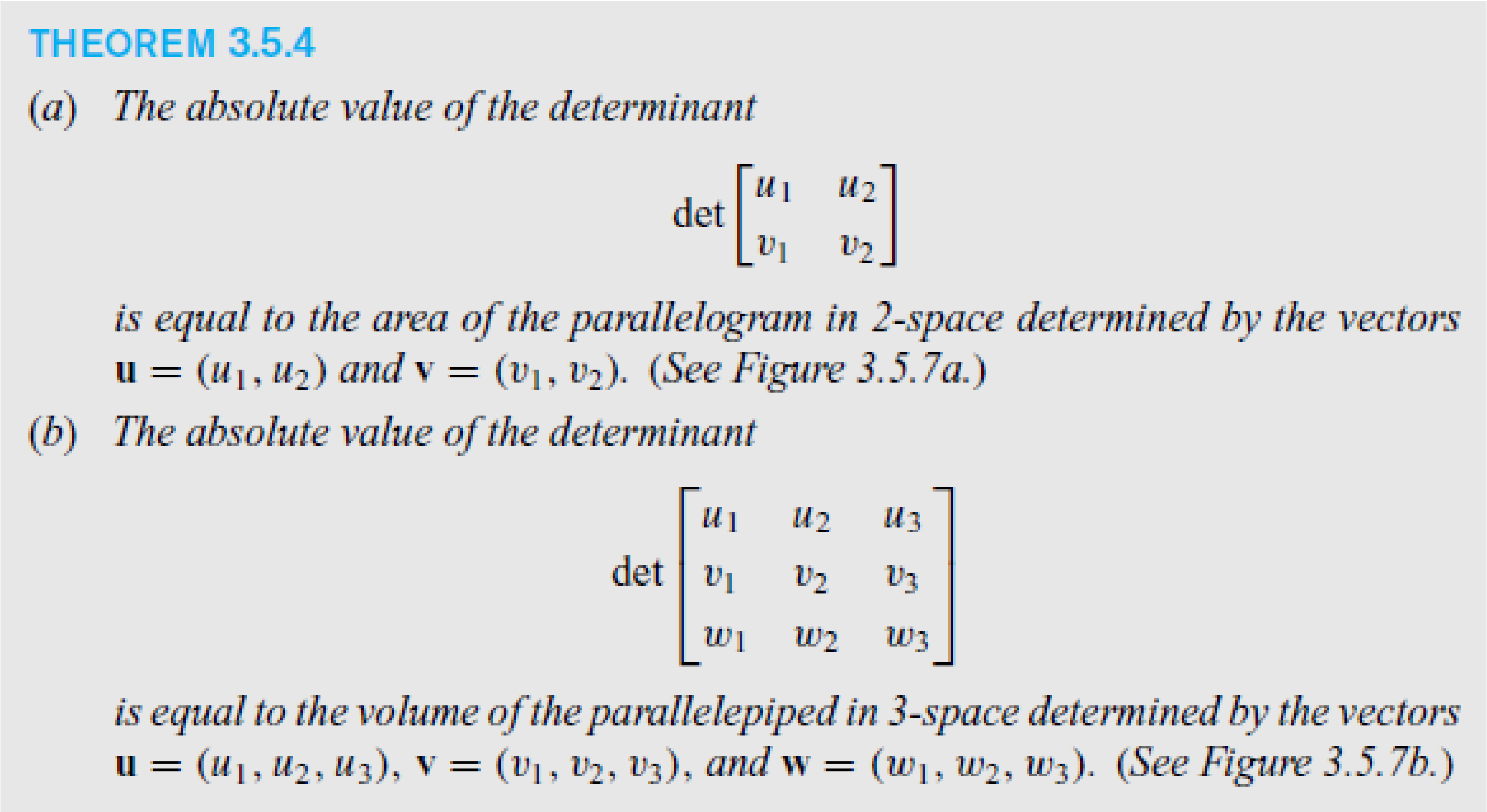


$$\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\| \sin \theta$$

THEOREM 3.5.3 Area of a Parallelogram

If $\mathbf{u}$ *and* $\mathbf{v}$ *are vectors in 3-space, then* $\|\mathbf{u} \times \mathbf{v}\|$ *is equal to the area of the parallelogram determined by* $\mathbf{u}$ *and* $\mathbf{v}$.

Geometry of Determinants



THEOREM 3.5.4

(*a*) *The absolute value of the determinant*

$$\det\begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}$$

is equal to the area of the parallelogram in 2-space determined by the vectors $\mathbf{u} = (u_1, u_2)$ *and* $\mathbf{v} = (v_1, v_2)$. *(See Figure 3.5.7a.)*

(*b*) *The absolute value of the determinant*

$$\det\begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}$$

is equal to the volume of the parallelepiped in 3-space determined by the vectors $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$, *and* $\mathbf{w} = (w_1, w_2, w_3)$. *(See Figure 3.5.7b.)*